Chapter 6

Multiple Integral

6.1 Introduction

Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function (on the three dimensional Cartesian plane where $z = f(x, y)$) and the plane which contains its domain. (Note that the same volume can be obtained via the triple integral. If there are more variables, a multiple integral will yield hypervolumes of multi-dimensional functions.

Multiple integration of a function in n variables: $f(x_1, x_2, ..., x_n)$ over a domain D is most commonly represented by nested integral signs in the reverse order of execution (the leftmost integral sign is computed last), followed by the function and integrand arguments in proper order (the integral with respect to the rightmost argument is computed last). The domain of integration is either represented symbolically for every argument over each integral sign, or is abbreviated by a variable at the rightmost integral sign. In this chapter, we will discuss line, double, triple integrals.

6.2 Line integral

In mathematics, a line integral (sometimes called a path integral, contour integral, or curve integral; not to be confused with calculating arc length using integration) is an integral where the function to be integrated is evaluated along a curve. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighing distinguishes the line integral from simpler integrals defined on intervals.

The line integral of f(x,y) along C is denoted by $\int_C f(x, y)\, ds$,

We use a ds here to acknowledge the fact that we are moving along the curve, C, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy). Because of the ds this is sometimes called the line integral of f with respect to arc length such that ds = $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$.

Parametric form of line segment joining 2 points (x_0, y_0, z_0) to (x_1, y_1, z_1) is expressed by:

$$x = (1-t)\,x_0 + t\,x_1,\ y = (1-t)\,y_0 + t\,y_1\,,\ z = (1-t)\,z_0 + t\,z_1,\ 0<t<1$$

Line integral of vector field

In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\overline{F}(x,y,z) = P(x,y,z)\overline{i} + Q(x,y,z)\overline{j} + R(x,y,z)\overline{k}$$

And the three-dimensional, smooth curve given by

$$\overline{r}(t) = x(t)\overline{i} + y(t)\overline{j} + z(t)\overline{k}\,, \quad a \le t \le b$$

Then the line integral of $\overline{F}$ along c is

$$\int_c \overline{F} \bullet d\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \bullet \overline{r'}(t)\,dt$$

That really is a dot product of the vector field and the differential and the differential really is a vector. Also

$$\overline{F}(\overline{r}(t)) = \overline{F}(x(t), y(t), z(t))$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int_c \overline{F} \bullet d\overline{r} = \int_a^b \overline{F} \bullet \overline{T}\, ds$$

Where $\overline{T}(t)$ is the unit vector and is given by $\overline{T}(t) = \dfrac{\overline{r'}(t)}{\left\| \overline{r'}(t) \right\|}$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above

$$\int_c \overline{F} \bullet d\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \bullet \frac{\overline{r'}(t)}{\left\| \overline{r'}(t) \right\|} \left\| \overline{r'}(t) \right\| dt = \int_a^b \overline{F}(\overline{r}(t)) \bullet \overline{r'}(t) dt$$

Example 1

Evaluate $\int_c \overline{F} \bullet d\overline{r}$, where $\overline{F}(x,y,z) = 8x^2yz\,\overline{i} + 5z\,\overline{j} - 4xy\overline{k}$ and c is the curve given by $\overline{r}(t) = t\,\overline{i} + t^2\,\overline{j} + t^3\overline{k}$, $0 \le t \le 1$.

Solution:

We first need the vector field evaluated along the curve.

$\overline{F}(\overline{r}(t)) = 8t^7\,\overline{i} + 5t^3\,\overline{j} - 4t^3\,\overline{k}$, $\overline{r'}(t) = \overline{i} + 2t\,\overline{j} + 3t^2\,\overline{k}$, then

$\overline{F}(\overline{r}(t)) \bullet \overline{r'}(t) = 8t^7 + 10t^4 - 12t^5$, therefore:

$$\int_c \overline{F} \bullet d\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \bullet \overline{r'}(t)\,dt = \int_0^1 [8t^7 + 10t^4 - 12t^5]\,dt = 1$$

Example 2

Evaluate $\int_c xy^4\,ds$, c is the right half of the circle $x^2 + y^2 = 16$ rotated in the counter clockwise direction.

Solution:

Since the parametric equation of the circle is x = 4cost, y = 4sint, then $\frac{dx}{dt}$ = -4sint and $\frac{dy}{dt}$ = 4cost, $-\pi/2 < t < \pi/2$, therefore $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 16$ and ds = 4 dt.

The line integral is then $\int_c xy^4 \, ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 \, (4)dt$ = 8192/5.

The line integral can be denoted by $\int_c P(x,y)\, dx + Q(x, y)\, dy$. To integrate this form, we have to express it in one variable (x or y or parametric equation) using the contour equation.

Example 3

Evaluate $\int_c \sin(\pi y)\, dx + yx^2 dy$, c is line segment from (0, 2) to (1, 4).

Solution:

Since the contour integration is the line joining (0,2) and (1,4) such that y = 2x + 2, therefore dy = 2dx.

Hence $\int_c \sin(\pi y)\,dx + yx^2 dy = \int_0^1 \sin(2\pi x)\,dx + (2x+2)x^2(2)dx$

$$= \int_0^1 \sin(2\pi x)\,dx + 4(x^3 + x^2)dx = \frac{-\cos(2\pi x)}{2\pi} + (x^4 + \frac{4}{3}x^3)\Big|_0^1 = \frac{7}{3}$$

Example 4

Evaluate $\int_c \sin(\pi y)\,dx + yx^2 dy$, c is line segment from (1,4) to (0,2)

Solution:

Since the contour integration is the line joining (1,4) and (0,2) such that y = 2x + 2, therefore dy = 2dx.

Hence $\int_c \sin(\pi y)\,dx + yx^2 dy = \int_1^0 \sin(2\pi x)\,dx + (2x+2)x^2(2)dx$

$$= \int_{1}^{0} [\sin(2\pi x) + 4(x^3 + x^2)]dx = \frac{-\cos(2\pi x)}{2\pi} + (x^4 + \frac{4}{3}x^3)\Big|_{1}^{0} = \frac{-7}{3}$$

From above example we note that

$$\int_{c} P(x,y)\, dx + Q(x, y)\, dy = -\int_{-c} P(x,y)\, dx + Q(x, y)\, dy$$

Example 5

Evaluate $\int_{c} y\, dx + x\, dy + z\, dz$, c is given by x = cost, y = sint, $z = t^2$, where $0 < t < 2\pi$.

Solution:

Since dx = -sint dt, dy = cost dt, dz = 2t dt, therefore:

$$\int_{c} y\, dx + x\, dy + z\, dz = \int_{0}^{2\pi} (\text{sint } (\text{-sint}) + \text{cost } (\text{cost}) + t^2(2t))\, dt$$

$$= \int_{0}^{2\pi} (\cos 2t + 2t^3)\, dt = (\frac{\sin 2t}{2} + \frac{t^4}{2})\Big|_{0}^{2\pi} = 8\pi^4$$

In any region where P dx + Q dy is independent of the path, the partial derivatives of the function

$$\phi(x, y) = \int_{(a,b)}^{(x,y)} P(x,y)\, dx + Q(x,y)\, dy$$

are $\frac{\partial \phi}{\partial x} = \phi_x = P(x,y)$ & $\frac{\partial \phi}{\partial y} = \phi_y = Q(x,y)$.

Theorem

If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at all points of a simply connected region R, then the integral $\int P(x,y)\,dx + Q(x,y)\,dy$ is independent on the path and conversely.

Example 6

Evaluate the integral $\int_{(-1,0)}^{(2,3)} (3x^2+3y)\,dx + (3x-4y)\,dy$, along the path $y^2 - 2x^2y + x^4 - 1 = 0$.

Solution:

Since $\frac{\partial Q}{\partial x} = 3 = \frac{\partial P}{\partial y}$, $Q(x, y) = 3x - 4y$, $P(x, y) = 3x^2+3y$, therefore the integration independent on the path, so we can consider the line $y = x+1$ is our path such that $dy = dx$ and

$$\int_{(-1,0)}^{(2,3)} (3x^2+3y)\,dx + (3x-4y)\,dy$$

$$= \int_{-1}^{2} (3x^2 + 3(x+1))\, dx + (3x - 4(x+1))\, dx$$

$$= \int_{-1}^{2} (3x^2 + 2x - 1)\, dx = 9$$

Problems

Evaluate the following integrals

1- $\int_c (xy + \ln x)\, dy$, c is the arc of parabola y = x^2 from (1,1) to (3,9)

2- $\int_c (x + 2y)\, dx + (x - y)\, dy$, c is the curve given by x = 2cost, y = 4sint

3- $\int_c 4x^3\, ds$, c is the line segment from (-2,-1) to (1,2)

4- $\int_c x\, ds$, c is the curve given by y = x^2, -1< x < 1

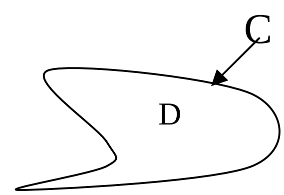
5- $\int_c xyz\, ds$, c is the helix given by $\bar{r}$ = (cost, sint, 3t), 0≤ t≤ 4π

6- $\int_c \overline{F} \bullet d\overline{r}$, where $\overline{F}(x,y,z) = xz\,\bar{i} - yz\,\bar{k}$ and c is the line segment from (-1,2,0) and (3,0,1)

##

6.3 Green theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals. Let's start off with a simple (recall that this means that it doesn't cross itself) closed curve C and let D be the region enclosed by the curve. Here is a sketch of such a curve and region.



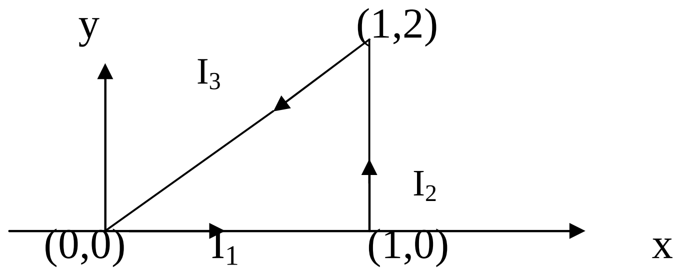
Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Example 7

Use Green's Theorem to evaluate $\int_c xy\,dx + x^2y^3dy$, where c is the triangle whose vertices (0,0), (1,0), (1,2) with positive orientation

Solution



If we express the problem in line integral, we have to divide the path into three paths $I_1 : y = 0$, $I_2 : x = 1$, $I_3 : 2x = y$ so that $dy = 0$, $dx = 0$, $2dx = dy$ respectively.

For path I_1: $\int_{I_1} xy\,dx + x^2y^3dy = \int_0^1 xy\,dx + x^2y^3dy = 0$

For path I_2: $\int_{I_2} xy\,dx + x^2y^3dy = \int_0^2 y^3dy = 4$,

For path I_3: $\int_{I_3} xy\,dx + x^2y^3dy = \int_1^0 [x(2x) + x^2(2x)^3(2)]dx = -\dfrac{10}{3}$.

Therefore $\int_C xy\,dx + x^2y^3dy = I_1 + I_2 + I_3 = \frac{2}{3}$.

By using Green theorem

$$\int_C xy\,dx + x^2y^3dy = \iint_D (2xy^3 - x)\,dxdy = \int_{x=0}^{1}\int_{y=0}^{y=2x} (2xy^3 - x)\,dydx$$

$$= \int_{x=0}^{1} (8x^5 - 2x^2)dx = \frac{2}{3}$$

Example 8

Evaluate $\int_C y^3\,dx - x^3dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution:

By using Green theorem, $\int_C y^3dx - x^3dy = \iint_D (-3x^2 - 3y^2)dxdy$.

We have to express the problem in polar form: $x = r\cos\theta$, $y = r\sin\theta$, therefore $\iint_D (-3x^2 - 3y^2)dxdy = -3\int_0^{2\pi}\int_{r=0}^{2} r^3\,dr\,d\theta$

$$= -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = -24\pi$$

Example 9

Use Green's Theorem to find the area of a disk of radius a.

Solution:

Since the area of the disk A = $\frac{1}{2}\int_{c} xdy\text{-}ydx$, C is the circle of radius a, thus by using Green theorem, we get:

$$A = \frac{1}{2}\int_{c} xdy\text{-}ydx = \iint_{D} dxdy$$

We have to express the problem in polar form: x = a $\cos\theta$, y = a $\sin\theta$, thus $\iint_{D} dxdy = \int_{0}^{2\pi}\int_{r=0}^{a} r\, dr\, d\theta = \int_{0}^{2\pi} \frac{r^2}{2}\Big|_{0}^{a} d\theta = a^2\pi$

6.4 Double integral

In calculus of a single variable the definite integral $\int_{a}^{b} f(x)\,dx$, for f(x) $\geq$ 0 is the area under the curve f(x) from x= a to x = b. For general f(x) the definite integral is equal to the area above the x-axis minus the area below the x-axis. The definite integral can be extended to functions of more than one variable. Consider a function of 2 variables z = f(x,y). The definite integral is denoted by $\iint_R f(x,y)\,dA$, where R is the region of integration in the xy-plane. For positive f(x,y), the definite integral is equal to the volume under the surface, z = f(x,y) and above xy-plane for x and y in the region R.

To evaluate a double integral we do it in stages, starting from the inside and working out, using our knowledge of the methods for single integrals. The easiest kind of region R to work with is a rectangle.

To evaluate $\iint_R f(x,y)\,dxdy$, proceed as follows:

• Work out the limits of integration if they are not already known

• Work out the inner integral for a typical y

• Work out the outer integral

Example 10

Evaluate $\int\limits_{y=1}^{2}\int\limits_{x=0}^{3} (1+8xy)\,dxdy$

Solution: In this example the "inner integral" is $\int\limits_{x=0}^{3} (1+8xy)\,dxdy$ with y treated as a constant, then

$$\int\limits_{y=1}^{2}\left[x+\frac{8x^2y}{2}\right]_0^3 dy = \int\limits_{y=1}^{2} (3+36y)\,dy = \left[3y+\frac{36y^2}{2}\right]_1^2 = 57$$

Example 11

Evaluate $\int\limits_{0}^{\pi/2}\int\limits_{0}^{1} y\sin x\,dydx$

Solution:

$$\text{Integral}=\int\limits_{0}^{\pi/2}\left[\int\limits_{0}^{1} y\sin x\,dy\right]dx = \int\limits_{0}^{\pi/2}\left[\frac{y^2\sin x}{2}\right]_0^1 dx = \int\limits_{0}^{\pi/2}\left[\frac{\sin x}{2}\right] dx$$

$$= \left[\frac{-\cos x}{2}\right]_0^{\pi/2} = ½$$

Example 12

Find the volume of the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 2\}$.

Solution: The volume under any surface $z = f(x, y)$ and above a region R is given by $V = \iint_R f(x,y)\,dxdy$

In our case $V = \int_0^2\int_0^1 (4-x-y)\,dxdy = \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_0^1 dy$

$$= \int_0^2 \left(\frac{7}{2} - y\right) dy = \left[\frac{7y}{2} - \frac{y^2}{2}\right]_0^2 = 5$$

The double integrals in the above examples are the easiest types to evaluate because they are examples in which all four limits of integration are constants. This happens when the region of integration is rectangular in shape. In non-rectangular regions of integration the limits are not all constant so we have to get used to dealing with non-constant limits. We do this in the next few examples.

Example 13

Evaluate $\int_0^2 \int_{x^2}^{x} y^2 x \, dydx$

Solution:

$$\int_0^2 \int_{x^2}^{x} y^2 x \, dydx = \int_0^2 \left[\frac{y^3 x}{3} \right]_{y = x^2}^{y = x} dx = \int_0^2 \left[\frac{x^4}{3} - \frac{x^7}{3} \right] dx \;=\; \left[\frac{x^5}{15} - \frac{x^8}{24} \right]_0^2$$

$$= -\frac{128}{15}.$$

Example 14

Evaluate $\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \sin(\frac{y}{x}) \, dydx$

Solution: Recall from elementary calculus the integral $\int \cos(my)\, dy = \frac{1}{m} \sin(my)$ for m independent of y. Using this result,

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \sin(\frac{y}{x}) \, dydx \;=\; \int_{\pi/2}^{\pi} \left[\frac{1}{x} \frac{\sin(y/x)}{1/x} \right]_{y=0}^{x^2} dx = \int_{\pi/2}^{\pi} \sin(x) dx = 1$$

Example 15

Evaluate $\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}}\, dxdy$

Solution:

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} dxdy \;=\; \int_1^4 \left[\frac{e^{x/\sqrt{y}}}{1/\sqrt{y}}\right]_{x=0}^{x=\sqrt{y}} dy \;=\; \int_1^4 \left[\sqrt{y}\, e - \sqrt{y}\right]_{x=0}^{x=\sqrt{y}} dy \;=$$

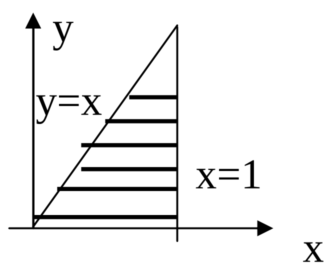
$$(e-1)\int_1^4 y^{1/2} dy \;=\; \frac{14}{3}(e-1).$$

When evaluating double integrals, it is very common not to be told the limits of integration but simply told that the integral is to be taken over a certain specified region R in the (x, y) plane. The integration can in principle be done in two ways: (i) integrating first with respect to x and then with respect to y, or (ii) first with respect to y and then with respect to x. The limits of integration in the two approaches will in general be quite different, but both approaches must yield the same answer. Sometimes one way round is considerably harder than the other, and in some integrals one way works fine while the other leads to an integral that cannot be evaluated using the simple methods you have been taught. There are no simple rules for deciding which order to do the integration in.

Example 16

Evaluate $\iint\limits_{D}(3-x-y)\,dA$ [dA means dxdy or dydx], where D is the triangle in the (x, y) plane bounded by the x-axis and the lines y = x and x = 1.

Solution:



Method 1

Do the integration with respect to x first. In this approach we select a typical y value which is (for the moment) considered fixed, and we draw a horizontal line across the region D; this horizontal line intersects the y axis at the typical y value. Find out the values of x (they will depend on y) where the horizontal line enters and leaves the region D (in this problem it enters at $x = y$ and leaves at $x = 1$). These values of x will be the limits of integration for the inner integral. Then you determine what values y has to range between so that the horizontal line sweeps the entire region D (in this case y has to go from 0 to 1). This determines the limits of integration for the outer integral, the integral with respect to y. For this particular problem the integral becomes

$$\iint_D (3-x-y)\,dA = \int_0^1\int_y^1 (3-x-y)\,dxdy = \int_0^1 \left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left[\frac{5}{2}-\frac{3y^2}{2}-4y\right] dy = \left[\frac{5y}{2}-2y^2+\frac{y^3}{2}\right]_0^1 = 1$$

Method 2

Do the integration with respect to y first and then x. In this approach we select a "typical x" and draw a vertical line across the region D at that value of x. Vertical line enters D at $y = 0$ and leaves at $y = x$. We then need to let x go from 0 to 1 so that the vertical line sweeps the entire region. The integral becomes

$$\iint_D (3-x-y)\, dA = \int_0^1 \int_{y=0}^{x} (3-x-y)\, dydx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

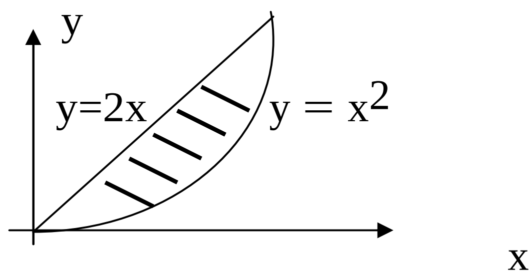
$$= \int_0^1 \left[3x - \frac{3x^2}{2} \right] dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1 = 1$$

Note that Methods 1 and 2 give the same answer. If they don't it means something is wrong.

Example 17

Evaluate $\iint_D (4x+2)\, dA$, where D is the region enclosed by the curves $y = x^2$ and $y = 2x$.

Solution: Again we will carry out the integration both ways, x first then y, and then vice versa, to ensure the same answer is obtained by both methods.



Method 1

We do the integration first with respect to x and then with respect to y. We shall need to know where the two curves $y = x^2$ and $y = 2x$ intersect. They intersect when $x^2 = 2x$, i.e. when $x = 0, 2$, so they intersect at the points $(0, 0)$ and $(2, 4)$.

For a typical y, the horizontal line will enter D at $x = y/2$ and leave at $x = \sqrt{y}$.

Then we need to let y go from 0 to 4 so that the horizontal line sweeps the entire region. Thus

$$\iint_D (4x+2)\, dA = \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} (4x+2)\, dxdy = \int_0^4 \Big[3x^2+2x \Big]_{x=y/2}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left[y+2\sqrt{y}-\frac{y^2}{2} \right] dy = \left[\frac{y^2}{2}+\frac{2y^{3/2}}{3/2}-\frac{y^3}{6} \right]_0^4 = 8.$$

Method 2

Integrate first with respect to y and then x, i.e. draw a vertical line across D at a typical x value. Such a line enters D at $y = x^2$ and leaves at y = 2x. The integral becomes

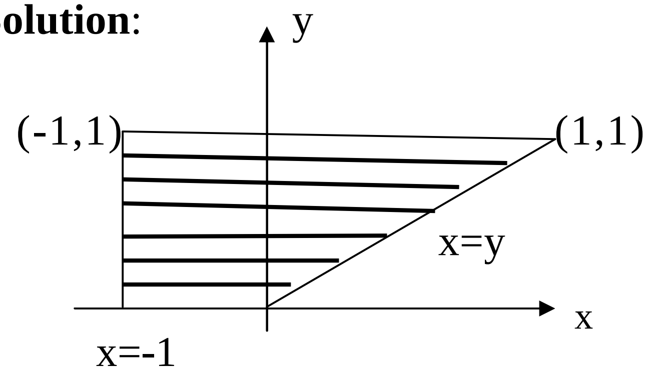
$$\iint_D (4x+2)\, dA = \int_0^2 \int_{y=x^2}^{y=2x} (4x+2)\, dydx = \int_0^2 \Big[4xy+2y\Big]_{y=x^2}^{y=2x} dx =$$

$$\int_0^2 \Big[6x^2-4x^3+4x\Big] dx = \Big[2x^3-x^4+2x^2\Big]_0^2 = 8.$$

Example 18

Evaluate $\iint_D (xy-y^3)\, dA$, where D is the region consisting of the square $\{(x, y): -1 \le x \le 0, 0 \le y \le 1\}$ together with the triangle $\{(x, y) : x \le y \le 1, 0 \le x \le 1\}$.

Solution:



Method 1

Integrate with respect to x first. A diagram will show that x goes from −1 to y, and then y goes from 0 to 1. The integral becomes

$$\iint_D (xy-y^3)\, dA = \int_0^1 \int_{x=-1}^{x=y} (xy-y^3)dxdy = \int_0^1 \left[\frac{x^2}{2} y - xy^3 \right]_{x=-1}^{x=y} dy =$$

$$\int_0^1 \left[-\frac{y^3}{2} - y^4 - \frac{y}{2} \right] dy = \left[-\frac{y^4}{8} - \frac{y^5}{5} - \frac{y^2}{4} \right]_0^1 = \frac{-23}{40}.$$

Method 2

It is necessary to break the region of integration D into two sub- regions D_1 (the square part) and D_2 (triangular part). The integral over D is given by

$$\iint_D (xy-y^3)\, dA = \iint_{D_1} (xy-y^3)\, dA + \iint_{D_2} (xy-y^3)\, dA$$

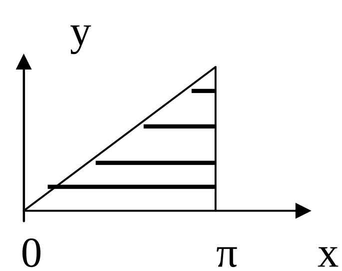
Thus the formula will be $\iint_D (xy-y^3)\, dA = \int_{-1}^{0}\int_{0}^{1}(xy-y^3)\, dydx + \int_{0}^{1}\int_{x}^{1}(xy-y^3)\, dydx = \int_{-1}^{0}\left[\frac{xy^2}{2}-\frac{y^4}{4}\right]_{y=0}^{y=1} dx + \int_{0}^{1}\left[\frac{xy^2}{2}-\frac{y^4}{4}\right]_{y=x}^{y=1} dx$

$$= \int_{-1}^{0}\left[\frac{x}{2}-\frac{1}{4}\right] dx + \int_{0}^{1}\left[\left[\frac{x}{2}-\frac{1}{4}\right]-\left[\frac{x^3}{2}-\frac{x^4}{4}\right]\right] dx = \left[\frac{x^2}{4}-\frac{x}{4}\right]_{-1}^{0} + \left[\frac{x^2}{4}-\frac{x}{4}-\frac{x^4}{8}+\frac{x^5}{20}\right]_{0}^{1} = -\frac{1}{2}-\frac{3}{40} = -\frac{23}{40}.$$

Example 19

Evaluate $\iint_D \frac{\sin x}{x}\, dA$, where D is the triangle $\{(x, y): 0 \le y \le x, 0 \le x \le \pi\}$.

Solution:



Let's try doing the integration first with respect to x and then y. This gives $\iint_D \frac{\sin x}{x}\,dA = \int_0^{\pi}\int_{x=y}^{\pi} \frac{\sin x}{x}\,dx\,dy$

but we cannot proceed because we cannot find an indefinite integral for $\frac{\sin x}{x}$. So, let's try doing it the other way. We then have $\iint_D \frac{\sin x}{x}\,dA = \int_0^{\pi}\int_{y=0}^{x} \frac{\sin x}{x}\,dy\,dx = \int_0^{\pi}\left[\frac{\sin x}{x}\,y\right]_{y=0}^{x} dx$

$= \int_0^{\pi} \sin x\,dx = \left[\cos x\right]_0^{\pi} = 2.$

Example 20

Consider the double integral $\iint_R (x^2 + xy^3)\,dA$, R is a rectangle $0 \le x \le 1$ and $1 \le y \le 2$.

Solution:

$$\iint_R (x^2 + xy^3)\,dA = \int_0^1 \left(\int_1^2 (x^2 + xy^3)\,dy\right) dx = \int_0^1 \left(x^2 y + \frac{xy^4}{4}\right)_1^2 dx$$

$$= \int_0^1 (x^2 + \frac{15x}{4})\, dx = (\frac{x^3}{3} + \frac{15x^2}{8})_0^1 = \frac{53}{24}.$$

Problems

Evaluate the following integrals

1- $\iint_R e^{-x-y}\, dxdy$, R is the region in the first quadrant in which $x+y \le 1$

2- $\iint_R (x^2 + y^2)dxdy$, R is the region $0 \le x \le y \le 1$

3- $\iint_R (x - y + 1)dxdy$, R is the region inside unit square in which $x+y \ge 0.5$.

4- $\iint_D (xy^3)dxdy$, D is the rectangle $0 \le x \le 2$, $0 \le y \le 1$

5- $\iint_D (x + y)dA$, D is the triangle with vertices (0,0), (0,2), (1,2)

6- $\iint_D (xy)dA$, D is the region bounded by the line $x + y = 4$ and xy coordinates.

7- $\iint\limits_D (48xy)dA$, where D is the region bounded by the line $y = x^3$ and $y = \sqrt{x}$.

8- $\iint\limits_D (x^2y)dA$, where D is the region bounded by the line $x+2y = 2$ and axes.

9- $\iint\limits_D (x+y^2)dA$, where D is the region given by $0 < x < 1$, $x < y < x+1$

10- $\iint\limits_D (yx^3)dA$, where D is the region given by 0<x<50-y< 50

11- $\int_1^2\int_0^y \frac{dxdy}{x^2+y^2}$, $\int_0^4 \int_0^{3\sqrt{16-x^2}/4} x\,dxdy$

12- $\iint\limits_D dxdy$ over the region bounded by $x \geq 0$, $y \geq 0$, $x + y \leq 1$

13- $\iint\limits_D xy\,dxdy$, where D is the region bounded by parabola $x = y^2$ and the lines $y = 0$ and $x + y = 2$ lying in the first quadrant.

##

We know how to change variables in a single integral:

$$\int_a^b f(x)\,dx = \int_a^b f(x(u))\,\frac{dx}{du}\,du$$

a & b are the new limits of integration. For double integrals the rule is more complicated. Suppose we have $\iint_D f(x,y)dxdy$ and want to change the variables to u and v given by x = x(u, v), y = y(u, v). The change of variables formula is: $\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v),y(u,v))|J|dudv$

Where J is the Jacobian given by $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

and D* is the new region of integration in the (u,v) plane.

A very commonly used substitution is conversion into polars. This substitution is particularly suitable when the region of integration D is a circle or an annulus (i.e. region between two concentric circles). Polar coordinates r and θ are defined by $x = r\cos\theta$, $y = r\sin\theta$

The variables u and v in the general description above are r and θ in the polar coordinates context and the Jacobian for polar coordinates is $\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$ = (cos θ)(rcos θ) - (-r sinθ)(sinθ) = r, so $|J| = r$ and the change becomes $\iint\limits_{D} f(x,y)\,dx\,dy = \iint\limits_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

Example 21

Use polar coordinates to evaluate $\iint\limits_{D} xy\,dx\,dy$, where D is the portion of the circle centre 0, radius 1 that lies in the first quadrant.

Solution: For the portion in the first quadrant we need 0 ≤ r ≤ 1 and 0 ≤ θ ≤ π/2. These inequalities give us the limits of integration in the r and θ variables, and these limits will all be constants. With x = r cosθ, y = r sinθ the integral becomes:

$$\iint\limits_{D} xy\,dx\,dy = \int_{0}^{\pi/2}\int_{r=0}^{1} r^2\cos\theta\sin\theta\,r\,dr\,d\theta = \int_{0}^{\pi/2}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^{1} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}\cos\theta\sin\theta \, d\theta = \int_0^{\pi/2} \frac{1}{8}\sin 2\theta \, d\theta = \frac{1}{8}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = 1/8$$

Example 22

Evaluate $\iint_D e^{-(x^2+y^2)} \, dxdy$, where D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: It is not feasible to attempt this integral by any method other than transforming into polars.

Let $x = r\cos\theta$, $y = r\sin\theta$. In terms of r and θ the region D between the two circles is described by $1 \le r \le 2$, $0 \le \theta \le 2\pi$, and so the integral becomes $\iint_D e^{-(x^2+y^2)} dxdy = \int_0^{2\pi}\int_1^2 e^{-r^2} r \, dr d\theta$

$$= \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_{r=1}^{2} d\theta = \int_0^{2\pi}\left[-\frac{1}{2}e^{-4} + \frac{1}{2}e^{-1}\right] d\theta = \pi(e^{-1} - e^{-4}).$$

Problems

Evaluate the following integrals by converting them into polar coordinates:

1- $\iint_D 2xy\, dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant

2- $\iint_D e^{-(x^2+y^2)}\, dxdy$, D is the unit circle centered at the origin.

3- $\int_0^1 \int_{r=0}^{\sqrt{1-y^2}} \cos(x^2+y^2)\ dxdy$

4- $\int_0^2 \int_{r=0}^{\sqrt{4-y^2}} (x^2+y^2)\ dxdy$

5- $\int_0^3 \int_{r=0}^{\sqrt{9-x^2}} \sqrt{(x^2+y^2)^3}\ dydx$

6- $\iint\limits_D (2x + 3y^2)\, dxdy$, D is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

##

6.5 Stokes' theorem

Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\vec{F}$ be a vector field then,

$$\int\limits_C \vec{F} \bullet d\vec{r} = \iint\limits_S \text{curl } \vec{F} \bullet d\vec{S}$$

In this theorem note that the surface S can actually be any surface so long as its boundary curve is given by C. This is something that can be used to our advantage to simplify the surface integral on occasion.

Example 23

Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\overline{F} \bullet d\overline{S}$, where $\overline{F} = z^2\,\overline{i} - 3xy\,\overline{j} + x^3y^3\,\overline{k}$ and S is the part of z = 5 - x^2 - y^2 above the plane z = 1. Assume S is oriented upwards.

Solution:

The boundary curve C will be where the surface intersects the plane z =1 and so will be the curve 1 = 5 - x^2 - y^2, i.e. the curve is $x^2 + y^2 = 4$ at z =1, so the boundary curve will be the circle of radius 2 that is in the plane z = 1. The parameterization of this curve is, $\overline{r}(t) = 2\cos t\,\overline{i} + 2\sin t\,\overline{j} + \overline{k}$, $0 \le t \le 2\pi$.

The first two components give the circle and the third component makes sure that it is in the plane z = 1.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint_S \text{curl}\,\overline{F} \bullet d\overline{S} = \int_C \overline{F} \bullet d\overline{r} = \int_0^{2\pi} \overline{F}(\overline{r}(t)) \bullet \overline{r'}(t)\,dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field, where $\overline{F}(\overline{r}(t)) = \overline{i} - 12\cos t \sin t\, \overline{j} + 64\cos^3 t \sin^3 t\, \overline{k}$ and $\overline{r'}(t)$ = $-2\sin t\, \overline{i} + 2\cos t\, \overline{j}$, then $\overline{F}(\overline{r}(t)) \bullet \overline{r'}(t) =$ -2sint -24sint $\cos^2 t$, thus $\iint\limits_S \text{curl}\, \overline{F} \bullet d\overline{S}$ =

$$\int_0^{2\pi} \overline{F}(\overline{r}(t)) \bullet \overline{r'}(t)\, dt = \int_0^{2\pi} (-2\sin t - 24\sin t \cos^2 t)\, dt = 0$$

Stoke's theorem states that, under conditions normally met in practice, the circulation of a vector field around the boundary of an oriented surface in space in the directions counterclockwise with respect to the surface's unit normal vector field n equals the integral of the normal component of the curl of the field over the surface.

The circulation of $\overline{F} = M\,\overline{i} + N\overline{j} + P\overline{k}$ around the boundary of C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector n equals the integral of $\nabla \times \overline{F} \bullet \overline{n}$ over S, i.e $\int_c \overline{F} \bullet d\overline{r} = \iint_s \nabla \times \overline{F} \bullet \overline{n}\, dS$.

Example 24

Calculate the circulation of the field $\overline{F} = x^2\,\overline{i} + 2x\overline{j} + z^2\overline{k}$ around the curve C: the ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above.

Solution:

$$\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 2\,\overline{k}$$

Since it is in the xy-plane, then $\overline{n} = \overline{k}$ & $(\nabla \times \overline{F}) \bullet \overline{n} = 2$, therefore $\int_c \overline{F} \bullet d\overline{r} = \iint_s 2 \; dxdy$.

We are working with the ellipse $4x^2 + y^2 = 4$ or $x^2 + y^2/4 = 1$, so I will use the transformation $x = r\cos\theta$ and $y = 2r\sin\theta$ to transform this ellipse into a circle. I will also have to use the

Jacobian to find the integrating factor for this integral but the jacobian of the ellipse $x^2 / a^2 + y^2 / b^2 = 1$ whose parametric equation $x = ar\cos\theta$ and $y = br\sin\theta$ is **abr**, thus $J(r,\theta) = 2r$,

therefore $\int_c \overline{F} \bullet d\overline{r} = \iint_s 2\,dx\,dy = \int_0^{2\pi} \int_{r=0}^{1} 2(2r)\,dr d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi$

Theorem

For any line integral $\int_c P dx + Q\,dy + R\,dz$ over closed contour c of an oriented surface S, then $\int_c P\,dx + Q\,dy + R\,dz =$ $\iint_s [(R_y - Q_z)\,\overline{i} + (P_z - R_x)\overline{j} + (Q_x - P_y)\overline{k}] \bullet \overline{n}\,dS$.

Example 25

Use Stoke's Theorem to evaluate the line integral

$$\int_c (y + 2z)dx + (x + 2z)\,dy + (x + 2y)\,dz$$

Where C is the curve formed by intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x+2y+2z = 0$.

Solution:

Let S be the circle cut by the sphere from the plane. Find the coordinates of the unit vector $\overline{n}$ normal to the surface S, therefore $\overline{n} = \dfrac{\overline{i} + 2\overline{j} + 2\overline{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{\overline{i} + 2\overline{j} + 2\overline{k}}{3}$.

By referring to the above theorem, P = y+2z, Q = x+2z, R = x + 2y, thus $\nabla \times \overline{F} = \overline{j}$, and so

$$\int_c (y + 2z)\, dx + (x + 2z)\, dy + (x + 2y)\, dz = \iint_s \overline{j} \bullet \left(\frac{\overline{i} + 2\overline{j} + 2\overline{k}}{3}\right) dS$$

$$= \iint_s \frac{2}{3}\, dS$$

Since the sphere $x^2 + y^2 + z^2 = 1$ is centered at the origin and the plane x + 2y + 2z = 0 also passes through the origin, the cross section is the circle of radius 1. Hence the integral is $\iint_s \frac{2}{3} dS = \frac{2}{3} \pi (1)^2$.

Example 26

Use Stoke's Theorem to evaluate the line integral

$$\int_C (x + z)\, dx + (x - y)\, dy + x\, dz$$

C is the ellipse defined by the equation $x^2/4 + y^2/9 = 1$, $z = 1$

Solution:

Let the surface S be the part of the plane $z = 1$ bounded by the ellipse. Obviously that the unit normal vector is $\overline{n} = \overline{k}$. Since $P = x + z$, $Q = x - y$, $R = x$, so $\nabla \times \overline{F} = \overline{k}$, therefore $\int_C (x + z)\, dx + (x - y)\, dy + x\, dz = \iint_S \overline{k} \bullet \overline{k}\, dS = \iint_S dS$. The double integral in the latter formula is the area of the ellipse. Therefore, the integral is $\iint_S dS = 2\,(3)\,\pi = 6\pi$.

Problems

1-Let C be the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant, oriented counterclockwise, where $F(x, y, z) = (x + y^2, y + z^2, z + x^2)$. Use Stokes' theorem to compute $\int_c \overline{F} \bullet ds$.

2- Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented with normal vector pointing upwards, and let $F(x, y, z) = (yz, x^2z, xy)$. Use Stokes' theorem to evaluate $\iint_S (\nabla \times \overline{F}) \bullet dS$.

3- Show that the line integral $\int_c (yz)dx + (xz)\,dy + (xy)\,dz$ is zero along any closed contour C.

4-Use Stoke's Theorem to calculate the line integral $\int_c (z-y)dx + (x-z)\,dy + (y-x)\,dz$. The curve C is the triangle with the vertices A(2,0,0), B(0,2,0), D(0,0,2).

5- Let C be the curve defined by the parametric equations:

$x = 0$, $y = 2 + 2\cos t$, $z = 2 + 2\sin t$, $0 \le t \le 2\pi$. Use Stokes' theorem to evaluate $\int_C (x^2 e^{5z})dx + (x \cos y)\, dy + (3y)\, dz$.

6- Evaluate the circulation of around the curve C where C is the circle $x^2 + y^2 = 4$ that lies in the plane z= -3, oriented counterclockwise with $\overline{F} = y\,\overline{i} + xz^3\,\overline{j} - zy^3\,\overline{k}$.

##

6.6 Gauss' divergence theorem

Suppose V is a subset of R^n (in the case of n= 3, V represents a volume in 3D space) which is compact and has a piecewise smooth boundary S. If $\overline{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then we have:

$$\iiint_V (\nabla \bullet \overline{F})\, dV = \iint_S (\overline{F} \bullet \overline{n})\, dS$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is quite generally the boundary of V oriented by outward-pointing normals, and $\overline{n}$ is the outward pointing unit normal field of the boundary ∂V. (d**S** may be used as shorthand for $\overline{n}$ dS.) By the symbol within the two integrals it is stressed once more that ∂V is a closed surface. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary ∂V.

Corollaries

By applying the divergence theorem in various contexts, other useful identities can be derived.

Applying the divergence theorem to the product of a scalar function g and a vector field $\overline{F}$, the result is

$$\iiint_V [\overline{F} \bullet (\nabla g) + g(\nabla \bullet \overline{F})] \; dV = \iint_S g\overline{F} \bullet dS$$

A special case of this is $\overline{F} = \nabla f$ in which case the theorem is the basis for Green's identities.

- Applying the divergence theorem to the cross-product of two vector fields $\overline{F} \times \overline{G}$, the result is

$$\iiint_V [\overline{G} \bullet (\nabla \times \overline{F}) - \overline{F} \bullet (\nabla \times \overline{G})] \; dV = \iint_S \overline{F} \times \overline{G} \bullet dS$$

- Applying the divergence theorem to the product of a scalar function, f, and a non-zero constant vector, the following theorem can be proven.

$$\iiint_V \nabla f \quad dV = \iint_S f \; dS$$

Applying the divergence theorem to the cross-product of a vector field F and a non-zero constant vector, the following theorem can be proven.

$$\iiint_V (\nabla \times \overline{F})\, dV = \iint_S dS \times \overline{F}$$

Example 27

Verify the planar variant of the divergence theorem for a region R, with $\vec{F}(x,y) = 2y\ \vec{i} + 5x\ \vec{j}$, where R is the region bounded by the circle $x^2 + y^2 = 1$

Solution:

The boundary of R is the unit circle C that can be represented parametrically by: $x = \cos(s)$; $y = \sin(s)$ such that $0 \le s \le 2\pi$ where s units is the length arc from the point s = 0 to the point P on C, then the vector equation of C is C(s) = cos(s) $\bar{i}$ + sin(s) $\bar{j}$ and at a point P = (cos(s) , sin(s)) on C, thus $\bar{F}$ (x,y) = 2 sin(s) $\bar{i}$ + 5cos(s) $\bar{j}$, therefore

$$\int_C \bar{F}\bullet\bar{n}\,ds = \int_0^{2\pi}(2\sin(s)\,\bar{i}+5\cos(s)\,\bar{j})\bullet(\cos(s)\,\bar{i}+\sin(s)\,\bar{j})ds =$$

$$\int_0^{2\pi}(2\sin(s)\cos(s)+5\sin(s)\cos(s))\,ds = \int_0^{2\pi}7\sin(s)\cos(s)\,ds = 0.$$

Because M = 2y, $\frac{\partial M}{\partial x}=0$ and N = 2y, $\frac{\partial N}{\partial y}=0$, thus

$$\iint_R \operatorname{div} f\ dA = \iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dA = 0$$

Problems

1- Use the divergence theorem to evaluate $\iint_S(\bar{F}\bullet\bar{n})\,dS$

where $\bar{F}=xy\,\bar{i}-\frac{y^2}{2}\bar{j}+z\,\bar{k}$ and the surface consists of the three surfaces, $z = 4- 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.

[Hint: $\iint_S(\bar{F}\bullet\bar{n})\,dS = \iiint_V(\nabla\times F)\,dV = \int_0^{2\pi}\int_0^{1}\int_0^{4-3r^2} r\,dz\,dr\,d\theta$]

2- Use divergence theorem to evaluate $\iint_S (3x\,\bar{i} + 2y\,\bar{j}) \bullet d\bar{S}$, S is the sphere $x^2 + y^2 + z^2 = 9$.

3-Use divergence theorem to evaluate $\iint_S (y^2 z\,\bar{i} + y^3\,\bar{j} + xz\,\bar{k}) \bullet d\bar{S}$ S is the boundary of the cube defined by $-1 \le x \le 1$, $-1 \le y \le 1$, $0 \le z \le 2$.

4- Let R be the region in R^3 bounded by the paraboloid $x^2 + y^2 = z$ and the plane $z = 1$ and let S be the boundary of the region R, evaluate $\iint_S (y\,\bar{i} + x\,\bar{j} + z^2\,\bar{k}) \bullet d\bar{S}$.

[Hint: $\iint_S (y\,\bar{i} + x\,\bar{j} + z^2\,\bar{k}) \bullet d\bar{S} = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 2z\, r\, dz\, dr\, d\theta$]

##